# TEMPERATURE STRESSES DUE TO A HEAT SOURCE LOCATED ON ONE SIDE OF A STRAIGHT WEDGE

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A heat source acts on one side of a straight wedge with adiabatic boundaries. The temperature stresses are determined in terms of the displacement potential and Airy stress function.

We will consider the quasi-static plane-stress problem of the temperature stress distribution in a straight infinite wedge $x \geq 0$, $y \geq 0$ with adiabatic boundaries. We assume that the thermophysical properties of the material do not go beyond the limits of elasticity and do not depend on temperature. We further assume that external forces are not applied to the wedge; consequently,

$$\sigma_\varphi = \tau_{r\varphi} = 0 \quad \text{at} \quad \varphi = 0, \ \varphi = \Pi/2.$$

We will find the temperature stresses in the form of sums,

$$\sigma_\varphi = \sigma_\varphi^1 + \sigma_\varphi^2 = \frac{\partial^2}{\partial r^2}(F - 2G\Phi), \tag{1}$$

$$\sigma_r = \sigma_r^1 + \sigma_r^2 = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right)(F - 2G\Phi), \tag{2}$$

$$\tau_{r\varphi} = \tau_{r\varphi}^1 + \tau_{r\varphi}^2 = -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial \varphi}(F - 2G\Phi)\right], \tag{3}$$

where $\Phi$, $F$ are the displacement potential and Airy function, satisfying the equations

$$\Delta\Phi = (1+\mu)\alpha T, \quad \Delta\Delta F = 0.$$

The expression for the displacement potential is written in the form [1]

$$\Phi = (1+\mu)\alpha a \int_0^t T dt + t\Phi_1 + \Phi_0. \tag{4}$$

Here $\Phi_1$ is an arbitrary harmonic function and $\Phi_0$ is the displacement potential corresponding to the initial temperature. These functions are selected so that the stresses are finite at zero and disappear at infinity.

If the initial temperature of the wedge is equal to zero, then after a time t the temperature due to an instantaneous linear heat source located at the point $(x_0, 0)$ is equal to

$$T = \frac{q}{2\Pi\lambda}[\exp(-\rho_1^2) + \exp(-\rho_2^2)].$$

The displacement potential is determined from (4) and has the form

$$\Phi = \frac{A}{2}[\mathrm{Ei}(-\rho_1^2) + \mathrm{Ei}(-\rho_2^2) - \ln\rho_1^2\rho_2^2].$$

From (1)–(3) it follows that

$$\sigma_\varphi^1 + \sigma_r^1 = -A_1[\exp(-\rho_1^2) + \exp(-\rho_2^2)], \tag{5}$$

$$\sigma_\varphi^1 - \sigma_r^1 = A_1[f_2(\rho_1)(\rho_1^2 - m^2) + f_2(\rho_2)(\rho_2^2 - m^2)], \tag{6}$$

$$\tau_{r\varphi}^1 = -A_1\rho_0\sin\varphi[f_2(\rho_1)\times$$
$$\times(\rho - \rho_0\cos\varphi) - f_2(\rho_2)(\rho + \rho_0\cos\varphi)], \tag{7}$$

where

$$f_2(\rho) = \frac{1 - \exp(-\rho^2)(1+\rho^2)}{\rho^4} = \sum_{k=0}^{\infty}\beta_k\rho^{2k}, \quad \beta_k = \frac{(-1)^k(k+1)}{(k+2)!}.$$

The stresses at the wedge boundary $\varphi = 0$ are

$$\tau_{r\varphi}^1 = 0,$$

$$\sigma_\varphi^1 = \frac{A_1}{2}\{f_1(\rho+\rho_0) + f_1(\rho-\rho_0)\} = \frac{A_1}{2}\left\{\sum_{k=0}^{\infty}\alpha_k[(\rho+\rho_0)^{2k} + (\rho-\rho_0)^{2k}]\right\} = A_1\sum_{k=0}^{\infty}\omega_k\rho^{2k}.$$

The stresses at the wedge boundary $\varphi = \Pi/2$ are

$$\tau_{r\varphi}^1 = 0, \ \sigma_\varphi^1 = A_1\left\{f_1\left(\sqrt{\rho^2+\rho_0^2}\right) - 2\rho_0^2 f_2\left(\sqrt{\rho^2+\rho_0^2}\right)\right\} = A_1\sum_{k=0}^{\infty}(\alpha_k - 2\rho_0^2\beta_k)(\rho^2+\rho_0^2)^k = A_1\sum_{k=0}^{\infty}\chi_k\rho^{2k}.$$

Here

$$\chi_k = \sum_{m=k}^{\infty} C_m^k \rho_0^{2m-2k}(\alpha_m - 2\rho_0^2\beta_m),$$

$$\omega_k = \sum_{m=k}^{\infty} C_{2m}^{2k}\rho_0^{2m-2k}\alpha_m,$$

$$f_1(\rho) = f_2(\rho)\rho^2 - \exp(-\rho^2) = \sum_{k=0}^{\infty}\alpha_k\rho^{2k},$$

$$\alpha_k = \frac{(-1)^{k+1}(2k+1)}{(k+1)!}.$$

The Airy stress function F is found in the form [2]

$$F = \sum_{k=0}^{\infty}\rho^{2k+2}[B_k U_k(\varphi) + D_k V_k(\varphi)],$$

where the functions

$$U_k(\varphi) = \frac{(-1)^k}{2}[\cos 2k\varphi - \cos(2k+2)\varphi],$$

$$V_k(\varphi) = \frac{1}{2}[\cos 2k\varphi + \cos(2k+2)\varphi]$$

satisfy the conditions

$$U_k(0) = U_k'(0) = U_k'(\Pi/2) = 0,\ U_k(\Pi/2) = 1,$$
$$V_k(0) = 1,\ V_k(\Pi/2) = V_k'(0) = V_k'(\Pi/2) = 0.$$

From the boundary conditions $\sigma_\varphi^2 = -\sigma_\varphi^1$ at $\varphi = 0$, $\varphi = \Pi/2$ we determine the coefficients $B_k$ and $D_k$,

$$D_k = -A_1\omega_k/(2k+2)(2k+1),$$
$$B_k = -A_1\chi_k/(2k+2)(2k+1).$$

It is more convenient to consider the calculation formulas in the form of a sum and a difference of stresses,

$$\sigma_\varphi + \sigma_r = \sigma_\varphi^1 + \sigma_r^1 - A_1 \sum_{k=0}^{\infty} a_k \rho^{2k} \cos 2k\varphi, \quad (8)$$

$$\sigma_\varphi - \sigma_r = \sigma_\varphi^1 - \sigma_r^1 - A_1 \times$$
$$\times \sum_{k=0}^{\infty} \rho^{2k} \{ka_k \cos 2k\varphi + b_k(k+1)\cos(2k+2)\varphi\}, \quad (9)$$

$$\tau_{r\varphi} = \tau_{r\varphi}^1 - \frac{A_1}{2} \sum_{k=0}^{\infty} \rho^{2k} \times$$
$$\times \{ka_k \sin 2k\varphi + b_{k+1}\sin(2k+2)\varphi\}. \quad (10)$$

Here the stresses with superscript 1 are determined from

$$(5)-(7),\ a_k = [\omega_k + (-1)^k \chi_k]/(k+1),$$
$$b_k = [\omega_k + (-1)^{k+1}\chi_k]/(k+1).$$

The problem of the stress distribution if the heat source acts during time t is similarly solved. The temperature distribution function, the displacement potential and the Airy function have, respectively, the forms

$$T = \frac{q}{2\Pi\lambda}[\mathrm{Ei}(-\rho_1^2) + \mathrm{Ei}(-\rho_2^2),$$

$$\Phi = \frac{At}{2}[\mathrm{Ei}(-\rho_1^2)(1+\rho_1^2) + \mathrm{Ei}(-\rho_2^2)(1+\rho_2^2) +$$
$$+ \exp(-\rho_1^2) + \exp(-\rho_2^2) - (1+\rho_1^2)\ln\rho_1^2 - (1+\rho_2^2)\ln\rho_2^2],$$

$$F = -A_0 \sum_{k=0}^{\infty} \frac{\rho^{2k+2}}{(2k+2)(2k+1)}[\bar{\chi}_k U_k(\varphi) + \bar{\omega}_k V_k(\varphi)],$$

where

$$\bar{\chi}_k = \sum_{m=k}^{\infty} C_m^k \rho_0^{2m-2k}(\bar{\alpha}_m - 2\rho_0^2\bar{\beta}_m),$$

$$\bar{\omega}_k = \sum_{m=k}^{\infty} C_{2m}^{2k} \rho_0^{2m-2k}\bar{\alpha}_m,\quad \bar{\alpha}_m = \frac{(-1)^k(2k+1)}{(k+1)!},$$

$$\bar{\beta}_m = (-1)^{k+1}/(k+2)!,\ k = 1,\ 2 \ldots,\ \bar{\alpha}_0 = \gamma - 2,\ \bar{\beta}_0 = -1/2.$$

The stress equations (8)–(10) remain valid for this case too, if we set $A_1 = A_0$,

$$\sigma_\varphi^1 - \sigma_r^1 = A_0 \left\{ \frac{1-\exp(-\rho_1^2)}{\rho_1^2} + \frac{1-\exp(-\rho_2^2)}{\rho_2^2} - \right.$$
$$\left. -2 - m^2[\psi_2(\rho_1) + \psi_2(\rho_2)] \right\},$$

$$\sigma_\varphi^1 + \sigma_r^1 = A_0\{\mathrm{Ei}(-\rho_1^2) + \mathrm{Ei}(-\rho_2^2) - \ln\rho_1^2\rho_2^2 - 4\},$$

$$\tau_{r\varphi}^1 = -A_0\rho_0 \sin\varphi\{\psi_2(\rho_1)(\rho - \rho_0\cos\varphi) -$$
$$-\psi_2(\rho_2)(\rho + \rho_0\cos\varphi)\},$$

$$\psi_2(\rho) = \frac{1-\exp(-\rho^2) - \rho^2}{\rho^4},$$

$$a_k = \frac{\bar{\omega}_k + (-1)^k\bar{\chi}_k}{k+1},\quad b_k = \frac{\bar{\omega}_k + (-1)^{k+1}\bar{\chi}_k}{k+1}.$$

In these problems the stresses are expressed in terms of uniformly convergent series in powers of the reciprocal of the Froude number. The solutions obtained can also be extended to the case of plane strain. For this purpose it is necessary to change the corresponding constants and add the stress $\sigma_z$.

## NOTATION

T is the temperature; t is the time; $a$, $\lambda$, $\alpha$ are the thermal diffusivity, thermal conductivity and coefficient of linear expansion; (x, y), (r, $\varphi$) are the coordinates of point in polar and rectangular coordinate systems, for which the polar axis and center coincide with the x axis and origin; ($x_0$, 0) are the coordinates of source of intensity q; $n^2 = 1/4at$; $\rho^2 = n^2r^2$; $\rho_0^2 = n^2x_0^2$; $m^2 = 2\rho_0^2\sin^2\varphi$; $\rho_1^2 = (r^2 + x_0^2 - 2rx_0\cos\varphi)n^2$; $\rho_2^2 = (r^2 + x_0^2 + 2rx_0\cos\varphi)n^2$; $\sigma_\varphi$, $\sigma_r$, $\tau_{r\varphi}$ are the stresses in polar coordinate system; $\Phi$, F are the displacement potential and Airy stress function; G, $\mu$ are the modulus of elasticity and Poisson's ratio; $\mathrm{Ei}(-x) = \int_\infty^x \frac{\exp(-t)}{t}dt$ is the integro-exponential function; $C_k^m$ is the number of combinations of k elements taken m at a time; $A = |-q(1+\mu)\alpha a/\Pi\lambda$; $A_1 = q(1+\mu)\alpha G/\Pi\lambda t$; $A_0 = A_1 t$; $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$; $\gamma = 0.577\ldots$ Euler constant.

## REFERENCES


1. H. Parkus, Nonstationary Temperature Stresses [Russian translation], Fizmatgiz, 1963.
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